Non-superposition effects in the Dirichlet–Casimir effect

2010 J. Phys. A: Math. Theor. 43 235402

(http://iopscience.iop.org/1751-8121/43/23/235402)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 235402 (15pp) doi:10.1088/1751-8113/43/23/235402

Non-superposition effects in the Dirichlet–Casimir effect

C Ccapa Ttira, C D Fosco and E L Losada

Centro Atómico Bariloche and Instituto Balseiro, Comisión Nacional de Energía Atómica, R8402AGP Bariloche, Argentina

E-mail: fosco@cab.cnea.gov.ar

Received 29 January 2010, in final form 12 March 2010
Published 13 May 2010
Online at stacks.iop.org/JPhysA/43/235402

Abstract

We study non-superposition effects in the Dirichlet–Casimir interaction energy for N boundaries in d spatial dimensions, quantifying its departure from the case of an interaction where a superposition principle is valid. We first derive some general results about those effects, and then show that they become negligible only when the distances between surfaces are larger than the sizes of each individual surface. We consider different examples of this situation in one, two and three spatial dimensions. Finally, we present two examples, corresponding to highly symmetric configurations involving more than two surfaces. We show that, even though superposition is not valid, the total interaction energy may nevertheless be expressed as a sum of (non-Casimir) energies involving pairs of surfaces.

PACS numbers: 03.70.+k, 03.65.−w, 11.10.−z

1. Introduction

Among the many interesting and distinctive features of the Casimir effect [1], not the least important is the fact that the forces it produces between (more than two) conducting surfaces do not satisfy, in general, a superposition principle. In other words, when dealing with more than two objects, the interaction energy cannot be written as the sum of the interaction energies corresponding to all the possible object pairs. As a consequence, knowledge of the energy of a system before the addition of an extra surface may seem to be of little help, since there is no obvious way to include the extra interaction terms.

Quite apart from its theoretical interest, it should be noted that this kind of phenomenon may also be of practical relevance, since it could be helpful in some approximate calculation schemes, in situations where nonlinear effects are small. For example, if there exists a regime where superposition is approximately valid, one should expect the dominant term in the Casimir energy to be akin to a two-body interaction potential, albeit with a non-Coulombian

potential, plus corrections. Under some assumptions, those corrections can be, as we shall see, small perturbations.

Excellent articles exist about the evaluation of the Casimir interaction energy within the *TGTG* formula [2, 3] (see also [4]) approach, by applying different expansions [2, 3, 5]. We present here an analysis of the perturbative expansion in powers of $\mathcal{O}$, from the point of view of the non-superposition effects, for the case of N boundaries. We then argue that when those effects are small, a perturbative expansion naturally suggests itself. We then discuss and apply that approximation within the context of different examples. Finally, we consider an example where the non-superposition effects are quite relevant; to show that, due to the highly symmetric configuration chosen, the energy may be written in terms of a unique 'two-body function'. However, the result is not just a sum over pairs of that function.

This paper is organized as follows: in section 2 we first review the functional approach to the calculation of the Casimir energy; in section 3, we deal with the study of non-superposition effects, relating them to a perturbative expansion in section 4, with examples in $1+1$, $2+1$ and $3+1$ dimensions; in section 5 we consider special arrangements of identical surfaces which allow for a non-perturbative treatment and in section 6 we present our conclusions.

2. The method

In order to analyze properties of the Casimir energy, it is convenient to introduce one of its concrete representations. We shall use here the one that is based on the functional integral formalism introduced in [6–8].

What follows is a review of its main aspects, adapted to the problem at hand.

Denoting by $\mathcal{Z}[\{\Sigma^{(a)}\}]$ the Euclidean vacuum amplitude for a real massless scalar field φ in $d+1$ dimensions, in the presence of N Dirichlet surfaces $\Sigma^{(a)}$, $a = 1, \ldots, N$, the total vacuum energy E_0 may be written as follows:

$$E_0 = -\lim_{T\to\infty}\left\{\frac{1}{T}\log\frac{\mathcal{Z}[\{\Sigma^{(a)}\}]}{\mathcal{Z}_0}\right\}, \tag{1}$$

where T is the extent of the (imaginary) time interval, and $\mathcal{Z}_0$ is the free (no surfaces) vacuum amplitude. The role of the latter is just to fix the vacuum energy to zero when there are no surfaces. Indices like a, that number the surfaces, will not be subjected to the summation convention.

On the other hand, the vacuum amplitude may be written as a functional integral[1]:

$$\mathcal{Z}[\{\Sigma^{(a)}\}] = \int [\mathcal{D}\varphi]\, \mathrm{e}^{-S_0(\varphi)}, \tag{2}$$

where S_0 is the free Euclidean action, which for the case at hand reads $S_0(\varphi) = \frac{1}{2}\int \mathrm{d}^{d+1}x\,(\partial\varphi)^2$, and $[\mathcal{D}\varphi]$ denotes the path integral measure corresponding to a scalar field which satisfies Dirichlet boundary conditions on each surface $\Sigma^{(a)}$.

Following the method developed in [10, 11], we begin by noting that it is quite useful to write that measure in the equivalent way:

$$[\mathcal{D}\varphi] = \mathcal{D}\varphi \times \prod_{a=1}^{N} \delta_{\Sigma^{(a)}}[\varphi], \tag{3}$$

where we introduced a δ-functional for the field on each surface, and $\mathcal{D}\varphi$ is the measure corresponding to the unconstrained field.

[1] In our use of the functional integral formalism, we follow the approach and conventions of [9].

In what follows, we focus on the $d = 3$ case, although every step will have its analog for different numbers of dimensions. The changes to the final expressions required to deal with $d \neq 3$ are described in section 3.

Thus, assuming that $(\sigma^1, \sigma^2) \to \mathbf{y}^{(a)}(\sigma)$ ($\mathbf{y}^{(a)} \in \mathbb{R}^{(3)}$) is a parametrization of $\Sigma^{(a)}$, we introduce an auxiliary field $\xi^{(a)}(\tau, \sigma)$ to exponentiate each functional delta:

$$\delta_{\Sigma^{(a)}}[\varphi] = \int \mathcal{D}\xi^{(a)}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau \int \mathrm{d}^2\sigma \sqrt{g^{(a)}(\sigma)}\xi^{(a)}(\tau,\sigma)\varphi[\tau,\mathbf{y}^{(a)}(\sigma)]} \tag{4}$$

(no sum over a), where $g^{(a)}(\sigma) \equiv \det\left[g^{(a)}_{\alpha\beta}(\sigma)\right]$ ($\alpha, \beta = 1, 2$) is the determinant of the induced metric $g^{(a)}_{\alpha\beta}$ on the surface, and $\tau \equiv x_0$. In terms of the previous parametrization,

$$g^{(a)}_{\alpha\beta}(\sigma) = \frac{\partial \mathbf{y}^{(a)}(\sigma)}{\partial \sigma^\alpha} \cdot \frac{\partial \mathbf{y}^{(a)}(\sigma)}{\partial \sigma^\beta} \quad \text{(no sum over } a\text{)}. \tag{5}$$

Equation (4) guarantees the (necessary) reparametrization invariance on each surface, assuming that the auxiliary fields behave as scalars under those transformations.

Substituting (4) into (3), we are left with an equivalent functional integral expression for $\mathcal{Z}[\{\Sigma^{(a)}\}]$:

$$\mathcal{Z}[\{\Sigma^{(a)}\}] = \int \left(\prod_{a=1}^{N} \mathcal{D}\xi^{(a)}\right) \int \mathcal{D}\varphi \exp\left\{-S_0(\varphi) + \mathrm{i}\int \mathrm{d}^4x\, J(x)\varphi(x)\right\}, \tag{6}$$

where we introduced $J(x) \equiv \sum_{a=1}^{N} J^{(a)}(x)$, with

$$J^{(a)}(x) = \int \mathrm{d}\tau\, \mathrm{d}^2\sigma \sqrt{g^{(a)}(\sigma)}\xi^{(a)}(\tau,\sigma)\delta(x_0 - \tau)\delta^{(3)}[\mathbf{x} - \mathbf{y}^{(a)}(\sigma)]. \tag{7}$$

Performing now the (Gaussian) integral over the φ field, the result may be put, in a condensed form, as follows:

$$\mathcal{Z}_\Sigma = \mathcal{Z}_0 \times \int \mathcal{D}\xi\, \mathrm{e}^{-S_\Sigma(\xi)} \tag{8}$$

where $\mathcal{Z}_\Sigma \equiv \mathcal{Z}[\{\Sigma^{(a)}\}]$, and $\mathcal{Z}_0 = \int \mathcal{D}\varphi\, \mathrm{e}^{-S_0(\varphi)}$.

In (8), $\mathcal{D}\xi$ denotes the integration measure for all the auxiliary fields (we assume there is more than one boundary) and S_Σ is a 'nonlocal action' for those fields:

$$S_\Sigma(\xi) = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^2\sigma \int \mathrm{d}\tau'\, \mathrm{d}^2\sigma' \sum_{a,b=1}^{N} [\xi^{(a)}(\tau,\sigma)\mathcal{M}_{(ab)}(\tau,\sigma;\tau',\sigma')\xi^{(b)}(\tau',\sigma')], \tag{9}$$

where each matrix element of $\mathcal{M}$ may be expressed in terms of the scalar field propagator $\mathcal{K}$:

$$\mathcal{M}_{(ab)}(\tau,\sigma;\tau',\sigma') = \sqrt{g^{(a)}(\sigma)}\mathcal{K}(\tau - \tau'; \mathbf{y}^{(a)}(\sigma) - \mathbf{y}^{(b)}(\sigma'))\sqrt{g^{(b)}(\sigma')}, \tag{10}$$

which, in $d = 3$, may be written as follows:

$$\begin{aligned} \mathcal{K}(x_0, \mathbf{x}) &= \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega x_0}\widetilde{\mathcal{K}}(\omega, \mathbf{x}), \\ \widetilde{\mathcal{K}}(\omega, \mathbf{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\omega^2 + \mathbf{k}^2} = \frac{\mathrm{e}^{-|\omega||\mathbf{x}|}}{4\pi|\mathbf{x}|}. \end{aligned} \tag{11}$$

Taking advantage of the time independence of the physical system considered, we Fourier transform in time the auxiliary fields, to write their action in the following way:

$$S_\Sigma = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \int \mathrm{d}^2\sigma \int \mathrm{d}^2\sigma' \sum_{a,b=1}^{N} [\tilde{\xi}^{(a)*}(\omega,\sigma)\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma,\sigma')\tilde{\xi}^{(b)}(\omega,\sigma')], \tag{12}$$

where the tilde on the fields denotes their corresponding Fourier transformed versions, and

$$\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma,\sigma') = \sqrt{g^{(a)}(\sigma)}\mathcal{K}_{(ab)}(\omega;\sigma,\sigma')\sqrt{g^{(b)}(\sigma')}, \tag{13}$$

where

$$\mathcal{K}_{(ab)}(\omega;\sigma,\sigma') \equiv \widetilde{\mathcal{K}}[\omega;\mathbf{y}^{(a)}(\sigma) - \mathbf{y}^{(b)}(\sigma')]. \tag{14}$$

Since the integral over the auxiliary fields is Gaussian, we have

$$\frac{\mathcal{Z}_\Sigma}{\mathcal{Z}_0} = \{\det[\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma,\sigma')\delta(\omega-\omega')]\}^{-\frac{1}{2}}, \tag{15}$$

where the determinant refers to the continuous indices $\omega, \omega', \sigma, \sigma'$, as well as the discrete ones a, b.

Then, recalling (1), we find for E_0 the expression [10, 11]

$$E_0 = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Tr}\log[\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma,\sigma')], \tag{16}$$

where the trace affects the σ, σ' and a, b indices (the trace over frequencies has been explicitly dealt with by means of the integral). We shall reserve the symbol 'tr' for the cases where a trace over just the (continuous) σ, σ' indices is needed.

Note that no subtraction of the would-be Casimir 'self-energies' has yet been performed; this step will be considered in the next section.

3. Non-superposition

Since we are interested in the Casimir *interaction* energy, we will first extract the self-energies of the surfaces. Besides, those energies are additive quantities, insensitive to the phenomenon we wish to consider.

That extraction can be done by factorizing a diagonal matrix constructed from the $a = b$ elements of $\widetilde{\mathcal{M}}$:

$$\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma,\sigma') = \int \mathrm{d}^2\sigma'' \sum_{c=1}^{N} \widetilde{\mathcal{D}}_{(ac)}(\omega;\sigma,\sigma'')\widetilde{\mathcal{T}}_{(cb)}(\omega;\sigma'',\sigma'), \tag{17}$$

where

$$\widetilde{\mathcal{D}}_{(ab)}(\omega;\sigma,\sigma') \equiv (\widetilde{\mathcal{M}}_{(aa)})(\omega;\sigma,\sigma')\delta_{ab} \tag{18}$$

(no sum over a). By construction, $\widetilde{\mathcal{T}}(\omega)$ has the matrix elements

$$\widetilde{\mathcal{T}}_{(ab)}(\omega;\sigma,\sigma') = \int \mathrm{d}^2\sigma''[\widetilde{\mathcal{M}}_{(aa)}]^{-1}(\omega;\sigma,\sigma'')\widetilde{\mathcal{M}}_{(ab)}(\omega;\sigma'',\sigma'), \tag{19}$$

(no sum over a). This factorization implies that $\det(\mathcal{M}) = \det(\mathcal{D})\det(\mathcal{T})$; thus, recalling (15) and (16), we may write

$$E_0 = \sum_{a=1}^{N} E_0^{(aa)} + E_I, \tag{20}$$

where

$$E_0^{(aa)} = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[\widetilde{\mathcal{M}}_{(aa)}(\omega)] \tag{21}$$

is the Casimir self-energy of the object labeled by the index a, and

$$E_I = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Tr}\log[\widetilde{\mathcal{T}}(\omega)]. \tag{22}$$

As already advanced, the self-energies, as seen from (20), are additive. Besides, they do not contribute to the Casimir forces between the surfaces, since they are independent of their relative distances.

Let us then consider the interaction term, E_I. It depends on $\widetilde{\mathcal{T}} \equiv \mathcal{I} + \widetilde{\mathcal{T}}'$ where $\mathcal{I}$ is the identity matrix (in both discrete and continuous indices) and $\widetilde{\mathcal{T}}'$ has vanishing diagonal ($a = b$) elements. Moreover, for $a \neq b$, it coincides with $\widetilde{\mathcal{T}}_{(ab)}$ of (19).

As a final step to obtain our main result, we derive a different (but equivalent) expression for $\widetilde{\mathcal{T}}_{(ab)}$, such that formula for the interaction energy does not contain explicit factors of the metric.

To that end, we introduce $G^{(a)}(\omega; \sigma, \sigma')$, the inverse of $\mathcal{K}_{(aa)}(\omega; \sigma, \sigma')$:

$$\int \mathrm{d}^2\sigma'' \mathcal{K}_{(aa)}(\omega; \sigma, \sigma'') G^{(a)}(\omega; \sigma'', \sigma') = \delta^{(2)}(\sigma - \sigma'). \tag{23}$$

Then,

$$[\widetilde{\mathcal{M}}_{(aa)}(\omega)]^{-1}(\sigma, \sigma') = \frac{1}{\sqrt{g^{(a)}(\sigma)}} G^{(a)}(\omega; \sigma, \sigma') \frac{1}{\sqrt{g^{(a)}(\sigma')}}, \tag{24}$$

and

$$\widetilde{\mathcal{T}}'_{(ab)}(\omega; \sigma, \sigma') = \frac{1}{\sqrt{g^{(a)}(\sigma)}} \mathcal{O}_{(ab)}(\omega; \sigma, \sigma') \sqrt{g^{(b)}(\sigma')}, \tag{25}$$

where

$$\widetilde{\mathcal{O}}_{(ab)}(\omega; \sigma, \sigma') \equiv \begin{cases} \int \mathrm{d}^2\sigma'' G^{(a)}(\omega; \sigma, \sigma'') \mathcal{K}^{(ab)}(\omega; \sigma'', \sigma') & \text{if} \quad a \neq b \\ 0 & \text{if} \quad a = b. \end{cases} \tag{26}$$

This way of writing $\mathcal{T}'_{(ab)}$ is rather convenient, since one can show that the determinants of the metric cancel, leading to a simpler final expression, depending only on $\mathcal{O}$:

$$E_I = \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} \operatorname{Tr} \log[\mathcal{I} + \mathcal{O}(\omega)], \tag{27}$$

which we use in our subsequent derivations. This is the N-body generalization [12] of the '*TGTG*' formula for the Casimir interaction between two bodies applied in [2, 3] which in our notation reads

$$E_I(\{\Sigma^{(1)}, \Sigma^{(2)}\}) = \frac{1}{2} \int \frac{\mathrm{d}\omega}{2\pi} \operatorname{tr} \log[1 - G^{(1)}(\omega) \mathcal{K}_{(12)}(\omega) G^{(2)}(\omega) \mathcal{K}_{(21)}(\omega)]. \tag{28}$$

The latter is obtained from (27) by expanding in powers of $\mathcal{O}$ and summing up the series for the particular case $N = 2$:

$$E_I(\{\Sigma^{(1)}, \Sigma^{(2)}\}) = -\int \frac{\mathrm{d}\omega}{2\pi} \sum_{k=1}^{\infty} \frac{1}{2k} \operatorname{tr}\{[\mathcal{O}_{(12)}(\omega) \mathcal{O}_{(21)}(\omega)]^k\}, \tag{29}$$

which yields (28).

Coming back to (27), we note that the form of $G^{(a)}$ shall depend, implicitly, on the geometry of each surface, and in general cannot be evaluated exactly, except in rather simple cases. However, most properties we shall deal with in this section are independent of that form.

Expression (27) has immediate analogs in $d \neq 3$. Indeed, in $d = 1$, we arrive to a result formally identical to (27), after one notes that the trace only affects the indices that label the 'surfaces', which in this case are just points labeled by their coordinates $x^{(a)}$ (no parameters σ are involved). Besides, the kernel $\widetilde{\mathcal{K}}(\omega; x)$ is now

$$\widetilde{\mathcal{K}}(\omega; x) = \frac{\mathrm{e}^{-|\omega||x|}}{2|\omega|}, \tag{30}$$

and $G^{(a)}$ becomes

$$G^{(a)}(\omega) = \left[\lim_{x,x'\to a} \frac{\mathrm{e}^{-|\omega||x|}}{2|\omega|}\right]^{-1} = 2|\omega|, \tag{31}$$

independent of a.

Finally, in $d = 2$ the boundaries are the curves $\Gamma^{(a)}$ described by just one parameter σ, and

$$\widetilde{\mathcal{K}}(\omega; \mathbf{x}) \equiv \frac{1}{2\pi} K_0(|\omega||\mathbf{x}), \tag{32}$$

where K_0 is a modified Bessel function of the second kind, and $G^{(a)}$ is obtained by evaluating the inverse of $\mathcal{K}_{(aa)}$, for which there is no general expression; we shall however derive its exact form for a particular case in the next section.

Equipped with (27), we can define a way to 'measure' the non-superposition effects. Again, we work in $d = 3$, but the results are straightforwardly adapted to $d \neq 3$. Assuming that we know $E_I\left(\{\Sigma_{(a)}\}_{a=1}^{N}\right)$, the energy corresponding to N surfaces, we add an extra boundary, $\Sigma_{(N+1)}$, obtaining a new energy $E_I\left(\{\Sigma_{(a)}\}_{a=1}^{N+1}\right)$.

If superposition were valid, the difference between the two energies would be the sum of the interaction energies between $\Sigma_{(N+1)}$ and $\Sigma_{(a)}$, with $a = 1, \ldots, N$. Thus, we introduce

$$\delta E_I(N) \equiv E_I\left(\{\Sigma_{(a)}\}_{a=1}^{N+1}\right) - E_I\left(\{\Sigma_{(a)}\}_{a=1}^{N}\right) - \sum_{a=1}^{N} E_I(\{\Sigma_{(N+1)}, \Sigma_{(a)}\}). \tag{33}$$

Superposition is broken whenever $\delta E_I(N) \neq 0$. Reciprocally, for the energy of N surfaces to verify superposition we would need $\delta E_I(M) = 0$, for $M = 2, \ldots, N-1$.

The final ingredient to evaluate $\delta E_I(N)$ is obtained by applying (27) to the $N+1$ surfaces. Then, we use determinant algebra to relate the determinant of the corresponding $(N+1)$th-order matrix to an Nth-order one:

$$\det[\mathcal{I}_{(ab)} + \mathcal{O}_{(ab)}]_{(N+1)\times(N+1)} = \det[\mathcal{I}_{(ab)} + \mathcal{O}'_{(ab)}]_{N\times N}, \tag{34}$$

where

$$\mathcal{O}'_{(ab)} \equiv \mathcal{O}_{(ab)} - \mathcal{O}_{(aN+1)}\mathcal{O}_{(N+1b)}. \tag{35}$$

Thus, the difference between the energies for $N+1$ and N surfaces may be put in the form

$$E_I\left(\{\Sigma_a\}_{a=1}^{N+1}\right) = E_I\left(\{\Sigma_a\}_{a=1}^{N}\right) + \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \operatorname{Tr}\log[\mathcal{I} - \mathcal{Q}], \tag{36}$$

where

$$\begin{aligned}\mathcal{Q}_{(ab)}(\omega; \sigma, \sigma') \equiv \int \mathrm{d}^2\sigma'' \int \mathrm{d}^2\sigma''' \sum_{c=1}^{N} \{&[(\mathcal{I} + \mathcal{O})^{-1}]_{(ac)}(\omega; \sigma, \sigma'') \\ &\times \mathcal{O}_{(cN+1)}(\omega; \sigma'', \sigma''')\mathcal{O}_{(N+1b)}(\omega; \sigma''', \sigma')\}.\end{aligned} \tag{37}$$

It is now a matter of algebra to extract the pair interaction energy to show that

$$\delta E_I(N) = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \operatorname{Tr}\log[I + \Lambda(\omega)] \tag{38}$$

with

$$\begin{aligned}\Lambda_{(ab)} = &[I - \mathcal{O}_{(N+1a)}\mathcal{O}_{(aN+1)}]^{-1} \\ &\times \left\{\sum_{c=1}^{N} [\mathcal{O}(\mathcal{I} + \mathcal{O})^{-1}]_{(ac)}\mathcal{O}_{(cN+1)}\mathcal{O}_{(N+1b)} - \mathcal{O}_{(aN+1)}\mathcal{O}_{(N+1b)} + \mathcal{O}_{(N+1a)}\mathcal{O}_{(aN+1)}\delta_{ab}\right\},\end{aligned} \tag{39}$$

where I in the first factor is the identity operator on functions defined in parameter space (while $\mathcal{I}$ also acts on the indices space), the discrete indices are not summed and the products are understood in the operatorial sense, regarding the kernels as matrix elements with continuous indices.

In spite of the fact that the form is rather complicated, we may already extract some conclusions from it. The most immediate one is that for the strength of the non-superposition effects to be small, the magnitude of the matrix elements of $\mathcal{O}$ between the $(N+1)$th surface and the previous ones has to be small.

Moreover, for the correction to be smaller than the superposition terms, we also need $\mathcal{O}_{(ab)}$, for $a, b = 1, \ldots, N$, to be small, since these operators also affect the magnitude of those terms. And this is the main conclusion of this section, namely that for superposition to be valid, all the matrix elements of $\mathcal{O}$ have to be small. We can see, in fact, that when that is the case, the *form* of the correction, to lowest order in the matrix elements, does depend on the matrix elements involving all the boundaries:

$$\delta E_I(N) \sim -\frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \sum_{a,b=1}^{N} \mathrm{tr}[\mathcal{O}_{(N+1a)}\mathcal{O}_{(ab)}\mathcal{O}_{(bN+1)}]. \tag{40}$$

The smallness of $\mathcal{O}$ is what, on the other hand, renders a perturbative expansion of the interaction energy possible.

4. Perturbative expansion

The condition that the matrix elements of $\mathcal{O}$ are small is precisely what one would require in order to expand the interaction energy in powers of that operator. On the other hand, for $\mathcal{O}$ to be small, the only assumption available here is that the N surfaces $\Sigma^{(a)}$ are compact objects, and that the distance between each pair of surfaces is much bigger than the size of any object. Under this assumption, the norm of $\mathcal{O}_{(ab)}$ is much smaller than 1, since the $G^{(a)}$ kernel is determined by the *inverse* of $\widetilde{\mathcal{K}}$ at small distances, while $\mathcal{K}_{(ab)}$ is, essentially, $\widetilde{\mathcal{K}}$ at long distances, and $\widetilde{\mathcal{K}}$ decreases with the distance.

The expansion yields a series for E_I:

$$E_I = \sum_{l=1}^{\infty} E_{I;l}, \tag{41}$$

where

$$E_{I;l} = \frac{(-1)^{l-1}}{2l}\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{Tr}[(\widetilde{\mathcal{O}}(\omega))^l], \tag{42}$$

or

$$E_{I;l} = \frac{(-1)^{l-1}}{2l}\int \frac{\mathrm{d}\omega}{2\pi} \sum_{a_1\neq a_2\neq a_3\neq \ldots \neq a_l \neq a_1} \mathrm{tr}\big[\widetilde{\mathcal{O}}_{(a_1a_2)}(\omega)\widetilde{\mathcal{O}}_{(a_2a_3)}(\omega)\ldots\widetilde{\mathcal{O}}_{(a_la_1)}(\omega)\big]. \tag{43}$$

This is, essentially, the long-distance expansion considered in [5], although we only deal with the Dirichlet (strong coupling) case.

It is worth noting at this point that the absence of explicit factors of the metric by no means signal a breaking of reparametrization invariance. Indeed, what happens is that the kernels denoted by $G^{(a)}$ do have non-trivial transformation properties under reparametrization, which compensate for the non-invariance of the integrals over the parameters.

Let us study the explicit form of the first few terms in this expansion. The $l = 1$ term vanishes, so that the lowest non-trivial order corresponds to $l = 2$, which using (25) becomes

$$E_{I;2} = -\frac{1}{4}\int \frac{\mathrm{d}\omega}{2\pi}\sum_{a\neq b}\int \mathrm{d}^2\sigma \int \mathrm{d}^2\sigma'\, \widetilde{\mathcal{O}}_{(ab)}(\omega;\sigma,\sigma')\widetilde{\mathcal{O}}_{(ba)}(\omega;\sigma',\sigma)$$
$$\equiv \sum_{a<b} E^{(ab)} \tag{44}$$

where

$$E^{(ab)} = -\frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\int_{\sigma,\sigma',\sigma'',\sigma'''} G^{(a)}(\omega;\sigma,\sigma')\mathcal{K}^{(ab)}(\omega;\sigma',\sigma'')$$
$$\times G^{(b)}(\omega;\sigma'',\sigma''')\mathcal{K}^{(ba)}(\omega;\sigma''',\sigma). \tag{45}$$

To this order, the total energy is obtained as the sum of 'interaction energies' corresponding to the pairs, in a sort of 'superposition principle'.

This property is violated in the next order term

$$E_{I;3} = \frac{1}{6}\int \frac{\mathrm{d}\omega}{2\pi}\sum_{a,b,c}\int \mathrm{d}^2\sigma \int \mathrm{d}^2\sigma' \int \mathrm{d}^2\sigma''\, \widetilde{\mathcal{O}}_{(ab)}(\omega;\sigma,\sigma')\widetilde{\mathcal{O}}_{(bc)}(\omega;\sigma',\sigma'')\widetilde{\mathcal{O}}_{(ca)}(\omega;\sigma'',\sigma)$$
$$\equiv \sum_{a<b<c} E^{(abc)} \tag{46}$$

where we have introduced a 'three-body energy interaction', $E^{(abc)}$:

$$E^{(abc)} = \int \frac{\mathrm{d}\omega}{2\pi}\widetilde{\mathrm{Tr}}[G^{(a)}(\omega)\mathcal{K}^{(ab)}(\omega)G^{(b)}(\omega)\mathcal{K}^{(bc)}(\omega)G^{(c)}(\omega)\mathcal{K}^{(ca)}(\omega)]. \tag{47}$$

Incidentally, this correction coincides with (40) when one considers $N+1$ surfaces, as it should be, since one should expect that the lowest-order violation to the non-superposition comes from the lowest non-quadratic term in the energy.

A fundamental ingredient in the calculation of the different terms in the expansion for E_I is the kernel $G^{(a)}(\omega;\sigma,\sigma')$. The form of that kernel depends strongly on the number of spatial dimensions as well as on the shape of the surface itself. Universal statements can only be made if more assumptions about the surfaces are made. However, based on the same assumption used to perform the series expansion, we may simplify the previous expressions further. Indeed, denoting by $\mathbf{x}^{(a)}$ the barycenter of the $\Sigma^{(a)}$ surface, we can, in the expressions above, use the approximation

$$\mathcal{K}^{(ab)}(\omega;\sigma',\sigma'') \simeq \widetilde{\mathcal{K}}(\omega;\mathbf{x}^{(a)} - \mathbf{x}^{(b)}). \tag{48}$$

This is justified by the following reason: we are assuming that $|\mathbf{x}^{(a)} - \mathbf{x}^{(b)}| \gg R^{(a)}, R^{(b)}$, where $R^{(a)}$ denotes the minimum radius for a sphere $S^{(a)}$, centered at $\mathbf{x}^{(a)}$, which encloses Σ_a. Then, we may replace $\mathbf{y}^{(a)}(\sigma) \to \mathbf{x}^{(a)}$ and $\mathbf{y}^{(b)}(\sigma') \to \mathbf{x}^{(b)}$, since $\mathcal{K}^{(ab)}(\omega;\sigma',\sigma'')$ is (under the previous assumptions) approximately constant inside $S^{(a)}$.

Using this approximation inside the expression for $E^{(ab)}$, we see that it may be written as follows:

$$E^{(ab)} \simeq \int \frac{\mathrm{d}\omega}{2\pi}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \rho^{(a)}(\omega;\mathbf{x})V(\omega;\mathbf{x}-\mathbf{y})\rho^{(b)}(\omega;\mathbf{y}), \tag{49}$$

where we introduced

$$\rho^{(a)}(\omega;\mathbf{x}) \equiv q_a(\omega)\delta^{(3)}(\mathbf{x}-\mathbf{x}^{(a)})$$
$$q_a(\omega) \equiv \int \mathrm{d}^2\sigma \int \mathrm{d}^2\sigma'\, G^{(a)}(\omega;\sigma,\sigma') \tag{50}$$

and

$$V(\omega; \mathbf{x}-\mathbf{y}) \equiv -\tfrac{1}{2}[\widetilde{K}(\omega; \mathbf{x}-\mathbf{y})]^2. \tag{51}$$

Thus, at this order, we see that the interaction energy for the a, b pair may be regarded as arising from integral over ω of the interaction energy for a set of point-like charges located at $\mathbf{x}^{(a)}$ and $\mathbf{x}^{(b)}$, whose strengths $q_a(\omega)$ and $q_b(\omega)$ are determined by the geometry of the respective surface.

On the other hand, the explicit form of the interaction potential is

$$V(\omega; \mathbf{x}-\mathbf{y}) \equiv -\frac{1}{2}\frac{\mathrm{e}^{-2|\omega||\mathbf{x}-\mathbf{y}|}}{(4\pi)^2|\mathbf{x}-\mathbf{y}|^2}; \tag{52}$$

hence, the interaction is always attractive. The integrals over $\mathbf{x}$ and $\mathbf{y}$ have been used in order to make it clear that each surface behaves as a sort of point-like charge. Of course, the same approximation may be used to simplify the form of the higher order terms.

The form of $G^{(a)}$ is not known exactly in general, except for particular situations like the $d = 1$ case, which we consider now.

4.1. $d = 1$

As a first test, we consider the case of two mirrors in $1 + 1$ dimensions. The operator $\mathcal{O}$ is just an ω-depending matrix, with matrix elements $\mathcal{O}_{(ab)}(\omega) = \mathrm{e}^{-|\omega||x^{(a)}-x^{(b)}|}$; the exponential decay assures the convergence of the perturbative expansion, regardless of the relative distances between the mirrors.

In this situation, the first (superposition) expression for the energy corresponding to two point-like objects (mirrors) located at $x^{(1)}$ and $x^{(2)}$ adopts the form

$$E^{(12)} = \int \frac{\mathrm{d}\omega}{2\pi}(2|\omega|)^2 V(\omega; x^{(1)} - x^{(2)}), \tag{53}$$

where

$$V(\omega; x^{(1)} - x^{(2)}) = -\frac{1}{2}\frac{\mathrm{e}^{-2|\omega||x^{(1)}-x^{(2)}|}}{(2|\omega|)^2}. \tag{54}$$

Assuming that the distance between the mirrors is a, we see that

$$E^{(12)} = -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-2|\omega|a} = -\frac{1}{4\pi a} = -\frac{0.07\,958}{a}, \tag{55}$$

to be compared with the exact result: $E = -\frac{\pi}{24a} \simeq -\frac{0.1309}{a}$, which is bigger by approximately 60%.

It is possible to calculate, for this case, all the higher order corrections exactly; only the even orders yield non-vanishing contributions, which are given by

$$E_{I;2l} = -\frac{1}{2l}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}(\mathrm{e}^{-2|\omega|a})^{2l} = -\frac{1}{4l^2\pi a}. \tag{56}$$

Then one sees that their sum

$$\sum_{l=1}^{\infty} E_{I;2l} = -\frac{1}{4\pi a}\sum_{l=1}^{\infty}\frac{1}{l^2} = -\frac{1}{4\pi a}\frac{\pi^2}{6} = -\frac{\pi}{24a}, \tag{57}$$

which is the exact result.

Besides, when more than two mirrors are considered, the energy becomes equal to the sum of the Casimir energies corresponding to the pairs formed by neighboring mirrors. For N mirrors, the expression for the energy is

$$E_I(N) = -\frac{\pi}{24}\sum_{a=1}^{N-1}\frac{1}{|x^{(a+1)} - x^{(a)}|}, \tag{58}$$

where we adopted a numbering such that $x^{(a)} > x^{(b)}$ for $a > b$. This result, in spite of its simple form, already exhibits non-superposition; indeed, substituting it into (33) yields an exact formula for $\delta E_I(N)$. When the $(N+1)$th mirror is at a position $x^{(N+1)} > x^{(N)}$:

$$\delta E_I(N) = \frac{\pi}{24}\sum_{a=1}^{N-1}\frac{1}{|x^{(N+1)} - x^{(a)}|}, \tag{59}$$

where it is evident that, for the magnitude of $\delta E_I(N)$ to be small, the new mirror has to be far from $x^{(a)}$, with $a = 1, \ldots, N-1$, although not necessarily far from $x^{(N)}$. An identical conclusion is drawn when the new mirror is put to the left of $x^{(1)}$. Finally, when the new mirror is inserted in between two mirrors, one can show that $\delta E_I(N)$ reaches a minimum if it is inserted at the midpoint between the two.

Note that the sign in (59) means that the Casimir energy is smaller (in absolute value) than it would be if superposition were valid.

4.2. $d = 2$

For a circle with radius R, using the angle ϕ as a parameter, we find

$$G^{(a)}(\omega; \phi, \phi') = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{\mathrm{e}^{in(\phi-\phi')}}{I_{|n|}(|\omega|R)K_{|n|}(|\omega|R)}, \tag{60}$$

where $I_{|n|}$ is a modified Bessel function of the first kind.

It is straightforward to check that one is in a situation of an $\mathcal{O}$ with small norm. Thus, in the perturbative expansion, when the surfaces are bounded and very far away, using the approximation that follows from

$$\mathcal{K}_{(ab)}(\omega; |\mathbf{x}^{(a)} - \mathbf{x}^{(b)}|) \sim \widetilde{\mathcal{K}}(\omega; |\mathbf{x}^{(a)} - \mathbf{x}^{(b)}|) \tag{61}$$

we may obtain an approximate expression for the case of N circles. Denoting by $\eta^{(a)}$ and $\mathbf{x}^{(a)}$ the radius and center of each circle, respectively, and assuming that $|\mathbf{x}^{(a)} - \mathbf{x}^{(b)}| \gg \max\{\eta^{(a)}\}$, we have for the pair interaction energy

$$E^{(ab)} \sim -\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\frac{[K_0(|\omega||\mathbf{x}^{(a)} - \mathbf{x}^{(b)}|)]^2}{I_0(|\omega|\eta^{(a)})K_0(|\omega|\eta^{(a)})I_0(|\omega|\eta^{(b)})K_0(|\omega|\eta^{(b)})}. \tag{62}$$

For the case of just two circles, 1 and 2, say, one can sum the series corresponding to the different powers of $\mathcal{O}$. The result is

$$E_I \sim -\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\log\left[1 - \frac{[K_0(|\omega||\mathbf{x}^{(a)} - \mathbf{x}^{(b)}|)]^2}{I_0(|\omega|\eta^{(a)})K_0(|\omega|\eta^{(a)})I_0(|\omega|\eta^{(b)})K_0(|\omega|\eta^{(b)})}\right]. \tag{63}$$

4.3. $d = 3$

We now deal with the case of surfaces in $d = 3$. The long-distance approximation requires the evaluation of the integral of $G^{(a)}$ over the parameters; for a case of a sphere, that integral is

$$q(\omega) = \frac{4|\omega|R^2}{I_{1/2}(|\omega|R)K_{1/2}(|\omega|R)}. \tag{64}$$

This is smaller than $\mathcal{K}_{(ab)}$ for distant surfaces, as a straightforward test shows.

If one assumes instead that the surfaces are really composed of small, weakly coupled surface elements, we may in fact use local approximations for the $G^{(a)}$ kernels. In this case, a local approximation means that the kernel is concentrated around $\sigma = \sigma'$:

$$G^{(a)}(\omega;\sigma,\sigma') \sim \eta^{(a)}(\omega,\sigma)\delta^{(2)}(\sigma-\sigma'), \tag{65}$$

where $a = 1, 2$, and $\eta^{(a)}$ will be determined now. Recalling that $G^{(a)}$ is defined as the inverse of $\mathcal{K}_{(aa)}(\omega;\sigma,\sigma')$, we explore the form of the latter in the neighborhood of a given point in the surface Σ_a, the one characterized by the parameter σ: $\mathbf{y}^{(a)}(\sigma)$. Close to that point, we derive the approximate expression

$$\begin{aligned}\mathcal{K}_{(aa)}(\omega;\sigma,\sigma') &\sim \int \frac{\mathrm{d}^2k_\parallel}{(2\pi)^2}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}_\parallel\cdot\partial_\alpha\mathbf{y}^{(a)}(\sigma)(\sigma^\alpha-\sigma'^\alpha)}}{2\sqrt{\mathbf{k}_\parallel^2+\omega^2}}\\ &\sim \frac{1}{4\pi\sqrt{g^{(a)}_{\alpha\beta}(\sigma)\delta\sigma^\alpha\delta\sigma^\beta}}\exp\bigl[-|\omega|\sqrt{g^{(a)}_{\alpha\beta}(\sigma)\delta\sigma^\alpha\delta\sigma^\beta}\bigr],\end{aligned} \tag{66}$$

where $\mathbf{k}_\parallel$ is the projection of the momentum along the tangent plane at the point $\mathbf{y}^{(a)}(\sigma)$, and $\delta\sigma^\alpha \equiv \sigma^\alpha - \sigma'^\alpha$.

In the assumption that there is no appreciable momentum flux between the different surface elements, we end up with the expression

$$\mathcal{K}_{(aa)}(\omega;\sigma,\sigma') \sim \frac{1}{2\sqrt{g^{(a)}(\sigma)}|\omega|}\delta^{(2)}(\sigma-\sigma'). \tag{67}$$

This yields

$$G^{(a)}(\omega;\sigma,\sigma') \sim 2\sqrt{g^{(a)}(\sigma)}|\omega|\delta^{(2)}(\sigma-\sigma') \quad\Rightarrow\quad \eta(\omega,\sigma) = 2|\omega|g^{(a)}(\sigma). \tag{68}$$

Let us first assume that we have two surfaces, $\Sigma^{(1)}$ and $\Sigma^{(2)}$, and consider the second-order expression for the interaction energy, using the local approximation for the kernels $G^{(1)}$ and $G^{(2)}$. We see that their interaction energy at this order becomes

$$E^{(12)} = -\frac{2^2}{2}\int\frac{\mathrm{d}\omega}{2\pi}\omega^2\int\mathrm{d}^2\sigma\sqrt{g^{(1)}(\sigma)}\int\mathrm{d}^2\sigma'\sqrt{g^{(2)}(\sigma')}[\mathcal{K}^{(12)}(\omega;\sigma,\sigma')]^2 \tag{69}$$

which, performing the integration over ω, results in the following expression:

$$E^{(12)} = \int\mathrm{d}^2\sigma\sqrt{g^{(1)}(\sigma)}\int\mathrm{d}^2\sigma'\sqrt{g^{(2)}(\sigma')}V(\sigma,\sigma'), \tag{70}$$

where

$$V(\sigma,\sigma') = -\frac{1}{32\pi^3}\frac{1}{|\mathbf{y}^{(1)}(\sigma)-\mathbf{y}^{(2)}(\sigma')|^5} \tag{71}$$

which looks like a kind of local-potential, van der Waals-like interaction.

Finally, let us consider the case of infinite parallel plates, within the quadratic approximation, using two different approaches. Obviously, in this case, the planes cannot be regarded as small surfaces and, even though the superposition approximation may be valid, certainly the planes cannot be regarded as point-like objects.

It is clear that, using as parameters the coordinates $\mathbf{x}_\parallel^{(a)}$ on each mirror, we have

$$G^{(a)}\bigl(\omega;\mathbf{x}_\parallel^{(a)},\mathbf{y}_\parallel^{(a)}\bigr) = \int\frac{\mathrm{d}^2k_\parallel}{(2\pi)^2}2\sqrt{k_\parallel^2+\omega^2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}_\parallel\cdot(\mathbf{x}_\parallel^{(a)}-\mathbf{y}_\parallel^{(a)})}, \tag{72}$$

while for $\mathcal{O}$ the result is

$$\mathcal{O}_{(ab)}\big(\omega;\mathbf{x}_\parallel^{(a)},\mathbf{y}_\parallel^{(b)}\big)=\int\frac{\mathrm{d}^2k_\parallel}{(2\pi)^2}\,\mathrm{e}^{-\sqrt{k_\parallel^2+\omega^2}|z^{(a)}-z^{(b)}|+\mathrm{i}\mathbf{k}_\parallel\cdot(\mathbf{x}_\parallel^{(a)}-\mathbf{y}_\parallel^{(a)})},\tag{73}$$

where $z^{(a)}$ is the position (on the third axis) of each plane. We see that, even in this case, the norm of the operator is small.

Indeed, substituting this into (45), we get for $\mathcal{E}_0$ the energy per unit area:

$$\mathcal{E}_0=-\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^2k_\parallel}{(2\pi)^2}\,\mathrm{e}^{-2a\sqrt{\mathbf{k}_\parallel^2+\omega^2}}=\frac{1}{16\pi^2a^3}\simeq-0.006\,33a^{-3},\tag{74}$$

to be compared with the exact result, that is $\mathcal{E}_{I;2}\simeq-0.006\,854a^{-3}$, which is a signal that the corrections are small.

If, on the other hand, we imagine each mirror as composed of weakly interacting infinitesimal surface elements (not a conductor), and apply the superposition result to a system composed of all the surface elements, then the energy per unit area to the first non-trivial order, $\mathcal{E}_{I;2}$, may be obtained by integrating the interaction energy between a single point on a mirror and all the points in the other. This corresponds to the following integral:

$$\mathcal{E}_{I;2}=4\int\frac{\mathrm{d}\omega}{2\pi}\omega^2\int\mathrm{d}^2x_\parallel\,V\big(\omega;\sqrt{a^2+\mathbf{x}_\parallel^2}\big).\tag{75}$$

The integral can be evaluated exactly, yielding

$$\mathcal{E}_{I;2}=-\frac{1}{24\pi^2a^3}\simeq-0.004\,22a^{-3}\tag{76}$$

which is different than the previously obtained result, as it corresponds to a different material.

5. Non-perturbative examples

Although we know that the exact N-body energy cannot be written as the sum of the energies of pairs of objects, that does not mean that there cannot be other less straightforward relations between those quantities.

As two concrete examples of that, we consider two highly symmetric situations, corresponding to a number of *identical* surfaces in two different spacial arrangements. Since the surfaces are assumed to be identical, we note that $G^{(a)}\equiv G^{(1)}$ is independent of a.

In the first case we consider, four surfaces are centered at the vertices of a regular tetrahedron, while in the second one they are in a cyclic arrangement.

5.1. Regular tetrahedron

We consider here four surfaces at the vertices of a regular tetrahedron. Note that the kernels $\mathcal{K}_{(ab)}$, connecting each surface to the other three, have the same form, $\mathcal{K}_{(12)}$, say. It is then quite straightforward to see that the eigenvalues of $\mathcal{O}$ are those of $\mathcal{O}_{(12)}$ times the ones of a 4×4 matrix m:

$$m=\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}.\tag{77}$$

Those eigenvalues are 3 and -1, the second appearing thrice. Then we have for the interaction energy

$$E_I = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I + 3\mathcal{O}_{(12)}] + \frac{3}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I - \mathcal{O}_{(12)}], \tag{78}$$

where we followed the convention that 'tr' denotes functional trace.

In other words, the energy is a sum of terms, each one of them essentially a two-body object. However, those two-body objects are not the ones one could expect from the 'real' surface pairs. Instead, they have inside geometrical factors, which come from the eigenvalues of m.

This should be compared to E_I^{sup}, the would-be superposition energy obtained by summing over the six (identical) Casimir energies corresponding to the pairs of surfaces. For each pair, the eigenvalues of $\mathcal{O}$ are now those of $\mathcal{O}_{(12)}$ times the ones of a 2×2 matrix m:

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{79}$$

Thus,

$$E_I^{\mathrm{sup}} = \frac{6}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I + \mathcal{O}_{(12)}] + \frac{6}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I - \mathcal{O}_{(12)}]. \tag{80}$$

The difference between E_I and E_I^{sup} is then

$$\begin{aligned} E_I - E_I^{\mathrm{sup}} = {} & \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I + 3\mathcal{O}_{(12)}] - \frac{3}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I - \mathcal{O}_{(12)}] \\ & - \frac{6}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log[I + \mathcal{O}_{(12)}], \end{aligned} \tag{81}$$

which is different from zero. One can check, however, by expanding in powers of $\mathcal{O}_{(12)}$ that the difference is at least of order 3 in that object (the first two orders vanish identically).

5.2. Cyclic arrangement

Now, the configuration is invariant under the rotation in a finite angle $\phi = \frac{2\pi}{N}$. This corresponds to an equilateral triangle for $N = 3$, and to regular polygons with N sides in the general case. The only approximation that we shall make is that the only relevant interaction terms in the matrix operator $\mathcal{O}$ are those corresponding to neighbors (for $N = 3$, this is of course unnecessary). The approximation should be more reliable (due to the fast decrease of the Casimir interaction) when $N \gg 1$, since then the distance from a surface to its next to nearest neighbor approaches twice the one to its nearest neighbor. The worst case regarding this approximation is $N = 4$, where that ratio is $\sqrt{2}$.

Thus, the kernels $\mathcal{K}_{(ab)}$ do not vanish only when the indices link consecutive surfaces, and all of them have the same form. Then the eigenvalues of $\mathcal{O}$ are those of $\mathcal{O}_{(12)}$ times $2\cos\left(\frac{2\pi\alpha}{N}\right)$, where $\alpha = 0, \ldots, N-1$. The interaction energy becomes

$$E_{I;N} = \frac{1}{2}\sum_{\alpha=0}^{N-1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log\left[I + 2\cos\left(\frac{2\pi\alpha}{N}\right)\mathcal{O}_{(12)}\right]. \tag{82}$$

In other words the energy becomes a sum

$$E_{I;N} = \sum_{\alpha=0}^{N-1}\mathcal{F}_\alpha \tag{83}$$

where

$$\mathcal{F}_\alpha = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{tr}\log\left[I + 2\cos\left(\frac{2\pi\alpha}{N}\right)\mathcal{O}_{(12)}\right] \tag{84}$$

again, each one of them essentially a two-body object. In this way, the Casimir energy becomes a sum involving the same object evaluated at different values of a constant α. This involves an approximation when $N > 3$, since one neglects some couplings, but the relation is exact for $N = 3$.

6. Conclusions

We have obtained an expression that measures the departure from superposition in the interaction Casimir energy corresponding to a massless scalar field in the presence of $N > 2$ Dirichlet surfaces. We have found that the most general condition under which the non-superposition effects can be regarded as small corresponds to a number of small surfaces separated by long distances. Under this assumption, one may construct a perturbative expansion, as a series in the operator $\mathcal{O}$.

The condition on that operator manifests itself in a different fashion, depending on the number of spatial dimensions. In $d = 1$, since the size of the mirrors is zero, one is in the best possible situation, namely the perturbative expansion is always reliable.

In $d = 2$ and $d = 3$, on the other hand, one can always obtain conditions under which the expansion should be reliable (although the rate of convergence depends on d).

An interesting conclusion one can extract is that the knowledge of the interaction energy for N surfaces is useful to calculate the one for $N + 1$ surfaces only when all the surfaces are widely separated.

We have also shown that, for some special cases, it is possible to write the energy as a sum of contributions, without making use of the perturbative expansion above. However, each term in the sum has a form that is not that of the energy for a pair, but nevertheless involves just two neighboring surfaces only, plus a geometrical coefficient. Similar properties might hold true also for different arrangements of identical surfaces, at least for configurations with a high degree of symmetry, and where the non-vanishing $\mathcal{O}$ matrix elements satisfy some non-trivial relations.

Acknowledgments

CCT, CDF and ELL thank CONICET, ANPCyT and UNCuyo for financial support.

References

[1] Plunien G, Müller B and Greiner W 1986 *Phys. Rep.* **134** 87
Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and Its Applications* (London: Clarendon)
Bordag M 1999 *The Casimir Effect 50 Years Later* (Singapore: World Scientific)
Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
Milton K A 2001 *The Casimir Effect: Physical Manifestations of the Zero-Point Energy* (Singapore: World Scientific)
Reynaud S *et al* 2001 *C. R. Acad. Sci. Paris* **IV-2** 1287
Milton K A 2004 *J. Phys. A: Math. Gen.* **37** R209
Lamoreaux S K 2005 *Rep. Prog. Phys.* **68** 201
Barrera R G and Reynaud S 2006 Focus on Casimir Forces *New J. Phys.* **8** (doi:10.1088/1367-2630/8/10/E05)
[2] Emig T, Graham N, Jaffe R L and Kardar M 2007 *Phys. Rev. Lett.* **99** 170403
[3] Kenneth O and Klich I 2008 *Phys. Rev.* B **78** 014103
[4] Balian R and Duplantier B 1978 *Ann. Phys.* **112** 165
[5] Milton K A and Wagner J 2008 *J. Phys. A: Math. Theor.* **41** 155402
[6] Li H and Kardar M 1992 *Phys. Rev.* A **46** 6490

[7] Emig T, Hanke A, Golestanian R and Kardar M 2001 *Phys. Rev. Lett.* **87** 260402
Emig T, Hanke A, Golestanian R and Kardar M 2003 *Phys. Rev.* A **67** 022114
[8] Emig T, Jaffe R L, Kardar M and Scardicchio A 2006 *Phys. Rev. Lett.* **96** 080403
Bordag M 2006 *Phys. Rev.* D **73** 125018
[9] Zinn-Justin J 2002 *Quantum Field Theory and Critical Phenomena* 4th edn (Oxford: Oxford Science Publications)
[10] Bordag M, Robaschik D and Wieczorek E 1985 *Ann. Phys.* **165** 192
[11] Robaschik D, Scharnhorst K and Wieczorek E 1987 *Ann. Phys.* **174** 401
[12] Bulgac A, Magierski P and Wirzba A 2006 *Phys. Rev.* D **73** 025007